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Quasi-Hermite–Fejér Type Interpolation of Higher Order

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1. INTRODUCTION

Let $0 \leqslant p \leqslant 3$ be a given integer and let

$$-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = +1 \tag{1}$$

be $n + 2$ distinct points. For a given function $f(x)$ continuous on $[-1, 1]$, the problem of Hermite–Fejér interpolation is to construct the polynomial of degree $2n + 3$ which interpolates $f(x)$ at these points with vanishing derivatives there. Recently P. Szasz [7] has introduced the notion of quasi-H–F interpolation when we interpolate $f(x)$ at all the nodes (1) but ask for vanishing derivatives at all but the first and last nodes. Following this idea, we shall call a polynomial $A_{n,p}(x) \equiv A_{n,p}(f; x)$ a quasistep parabola of order 3 if

$$\begin{aligned} A_{n,p}(x_k) &= f(x_k), \qquad k = 1,..., n, \\ A_{n,p}^{(\nu)}(-1) &= A_{n,p}^{(\nu)}(+1) = 0, \qquad \nu = 1,..., p - 1, \\ A_{n,p}^{(l)}(x_k) &= 0, \qquad k = 1,..., n, \qquad l = 1, 2, 3, \\ A_{n,p}(-1) &= f(-1), \qquad A_{n,p}(+1) = f(1) \qquad \text{if} \quad p \neq 0. \end{aligned} \tag{2}$$

If $p = 0$, the values and derivatives at -1 and $+1$ are not being considered. Szasz studied quasistep parabolas of order 1 and rediscovered a result of Egervary and Turán [1] on "stable" and "most economical" interpolation process.

The object of this note is to obtain quasi H–F type interpolation formulae of higher order when the nodes are the zeros of ultraspherical polynomials. Following the usual convention we denote by $T_n(x)$, $U_n(x)$ and $P_n(x)$ the Tchebycheff of first kind, of the second kind and the Legendre polynomials respectively. In the special case of the zeros of $T_n(x)$, $P_n(x)$, $U_n(x)$ and $P'_{n+1}(x)$, we study the positivity and convergence properties of the interpolatory polynomials with a suitable choice of p.

2. Explicit Form of $A_{n,p}(f; x)$

We shall consider only the case when $x_1, x_2, \ldots, x_n$ in (1) are the zeros of the ultraspherical polynomial $P_n^{(\lambda)}(x)$, $\lambda > -\frac{1}{2}$. Then

$$A_{n,p}(f; x) = \sum_{k=0}^{n+1} f(x_k)\, \lambda_k(x), \tag{2.1}$$

where $\{\lambda_k(x)\}_0^{n+1}$ are the fundamental polynomials of the interpolation problem (2). More precisely, the polynomials $\lambda_k(x)$ are of degree $4n + 2p - 1$ and are determined uniquely by the following conditions:

$$\lambda_k(x_j) = \delta_{jk}, \qquad j, k = 0, 1, \ldots, n + 1. \tag{2.1}$$

$$\lambda_k'(x_j) = \lambda_k''(x_j) = \lambda_k'''(x_j) = 0, \quad j = 1, \ldots, n, \quad k = 0, 1, \ldots, n + 1, \tag{2.2}$$

$$\lambda_k^{(l)}(-1) = \lambda_k^{(l)}(+1) = 0, \qquad l = 1, \ldots, p - 1. \tag{2.3}$$

If $p = 0$, (2.3) is void and in (2.1) the points x_0 and x_{n+1} are not considered. If $p = 1$, (2.3) is again void. We formulate

Theorem 1. *The polynomials $\lambda_k(x)$ in (2.1) for $k = 1, \ldots, n$ are given by*

$$\lambda_k(x) = l_k^4(x) \frac{(1 - x^2)^p}{(1 - x_k^2)^p} [1 + A_1(x - x_k) + A_2(x - x_k)^2 + A_3(x - x_k)^3], \tag{2.4}$$

where $\omega(x) = \prod_1^n (x - x_j)$,

$$l_k(x) = (\omega(x))/((x - x_k)\, \omega'(x_k)), \tag{2.5}$$

$$A_1 = 2(p - \nu)x_k/(1 - x_k^2),$$
$$A_2 = \frac{x_k^2}{(1 - x_k^2)^2} \{P + (p - \nu)^2\} + \frac{Q}{1 - x_k^2}, \tag{2.6}$$
$$A_3 = \frac{Bx_k^3}{(1 - x_k^2)^3} + \frac{Cx_k}{(1 - x_k^2)^2},$$

with

$$\nu = 2\lambda + 1, \qquad N = n^2 + 2n, \qquad Q = \tfrac{1}{3}(2N - 2\nu + 3p),$$
$$P = (p - \nu)^2 + 2(p - \nu) - \frac{\nu(\nu - 4)}{6},$$
$$B = \tfrac{1}{3}(p - \nu + 1)(2p - \nu)(2p - 3\nu + 4), \tag{2.7}$$
$$C = \tfrac{2}{3}N(2p - 2\nu + 1) + \tfrac{1}{6}(2p - \nu)(6p - 7\nu + 6).$$

Proof. We sketch the proof and omit the calculations. Since $\omega(x)$ satisfies the differential equation

$$(1 - x^2)\,\omega'' - \nu x\omega' + N\omega = 0, \tag{2.8}$$

it follows easily from (2.5) that

$$\begin{aligned} 2l_k'(x_k) &= \frac{\omega''(x_k)}{\omega'(x_k)} = \frac{\nu x_k}{1 - x_k^2}, \\ 3l_k''(x_k) &= \frac{\omega'''(x_k)}{\omega'(x_k)} = \frac{\nu(\nu+2)\,x_k^2}{(1 - x_k^2)^2} - \frac{N-\nu}{1-x_k^2}, \\ 4l_k'''(x_k) &= \frac{\omega^{(4)}(x_k)}{\omega'(x_k)} = \frac{\nu(\nu+2)(\nu+4)\,x_k^3}{(1-x_k^2)^3} - \frac{(\nu+2)(2N-3\nu)\,x_k}{(1-x_k^2)^2}. \end{aligned} \tag{2.9}$$

It follows from (2.1)–(2.3) that for $k = 1, 2, \dots, n$, the polynomial $\lambda_k(x)$ will have the formal expression (2.4). The values of A_1, A_2, A_3 are determined from the conditions

$$\lambda_k'(x_k) = \lambda_k''(x_k) = \lambda_k'''(x_k) = 0.$$

Remark. The explicit form of the polynomials $\lambda_0(x)$, $\lambda_{n+1}(x)$ have the following forms when $p \geqslant 1$:

$$\lambda_0(x) = \frac{(\omega(x))^4}{(\omega(1))^4}\left(\frac{x+1}{2}\right)^p \sum_{k=0}^{p-1} B_k(x-1)^k.$$

$$\lambda_{n+1}(x) = \left(\frac{\omega(x)}{\omega(-1)}\right)^4 \left(\frac{1-x}{2}\right)^p \sum_{k=0}^{p-1} C_k(1+x)^k.$$

The values of B_k, C_k depend on p and will be given later in Section 3 for $p = 0, 1, 2, 3$.

3. Some Special Cases of Positive Operators $\Lambda_{n,p}$

From Theorem 1, we can rewrite the polynomials $\lambda_k(x)$. In fact, we have

$$\lambda_k(x) = \frac{l_k^4(x)(1-x^2)^p}{(1-x_k^2)^p}\left[\left\{1 + \frac{(p-\nu)(x-x_k)\,x_k}{1-x_k^2}\right\}^2 + \frac{(x-x_k)^2}{(1-x_k^2)^2} L_k(x)\right] \tag{3.1}$$

where

$$L_k(x) = Px_k^2 + Q(1-x_k^2) + \left\{\frac{Bx_k^3}{1-x_k^2} + Cx_k\right\}(x - x_k), \tag{3.2}$$

and P, Q, B and C are given by (2.7).

In order that $\lambda_k(x) \geqslant 0$ for $|x| \leqslant 1$, $k = 1,..., n$ it is enough that $L_k(x) \geqslant 0$ for $|x| \leqslant 1$. Rearranging (3.2) we have

$$L_k(x) = Px_k^2 + \frac{(3p - 2\nu)}{3}(1 - x_k^2) + \frac{Bx_k^3}{1 - x_k^2}(x - x_k)$$
$$+ \tfrac{1}{6}(2p - \nu)(6p - 7\nu + 6)\, x_k(x - x_k)$$
$$+ \frac{2N}{3}\{1 - x_k^2 + (2p - 2\nu + 1)\, x_k(x - x_k)\}. \qquad (3.3)$$

For $|x_k| \leqslant 1 - \epsilon$, the last term in the above expression is the dominant part. If we choose p and ν such that

$$1 - x_k^2 + (2p - 2\nu + 1)\, x_k(x - x_k) \geqslant 0, \qquad -1 \leqslant x \leqslant 1, \qquad (3.4)$$

for $k = 1,..., n$, then $L_k(x) \geqslant 0$ for $|x| \leqslant 1$ and for $|x_k| \leqslant 1 - \epsilon$. A sufficient condition for (3.4) to hold is that

$$|p - \nu + 1| \leqslant \tfrac{1}{2}. \qquad (3.5)$$

For the sake of simplicity, we shall consider the case when $p - \nu + 1 = 0$. In this case from (2.7) it follows that $B = 0$ and the expression (3.3) becomes much simpler:

$$L_k(x) = \frac{\nu - 3}{3} - \frac{\nu(\nu - 2)}{6} x x_k + \frac{2N}{3}(1 - x x_k). \qquad (3.6)$$

Since p is a nonnegative integer $\leqslant 3$, the following four cases arise:

(i) $p = 0$, $\nu = 1$, $\lambda = 0$, $\omega(x) = T_n(x) = \cos n\theta$, $x = \cos\theta$

(ii) $p = 1$, $\nu = 2$, $\lambda = \frac{1}{2}$, $\omega(x) = P_n(x)$ with $P_n(1) = 1$

(iii) $p = 2$, $\nu = 3$, $\lambda = 1$, $\omega(x) = U_n(x) = \dfrac{\sin(n + 1)\,\theta}{\sin\theta}$, $x = \cos\theta$

(iv) $p = 3$, $\nu = 4$, $\lambda = \frac{3}{2}$, $\omega(x) = P'_{n+1}(x)$.

It is now easy to find the explicit form of the polynomials $\lambda_0(x)$ and $\lambda_{n+1}(x)$ in these four cases. As remarked just prior to Theorem 1, in case (i), $\lambda_0(x) = \lambda_{n+1}(x) = 0$. In case (ii), we have

$$\lambda_0(x) = \lambda_{n+1}(-x) = ((1 + x)/2)(P_n(x))^4. \qquad (3.7)$$

In case (iii), we have

$$\lambda_0(x) = \lambda_{n+1}(-x) = \frac{(1 + x)^2\{U_n(x)\}^4}{4(n + 1)^4}\left\{1 + \frac{(2n + 1)(2n + 3)}{3}(1 - x)\right\}. \qquad (3.8)$$

Lastly, in case (iv), we can verify that

$$\lambda_0(x) = \frac{(1+x)^3\{P'_{n+1}(x)\}^4}{\{(n+1)(n+2)\}^4}\Big\{2 + n(n+3)(1-x) + \frac{(n-3)(n+1)(n+2)(n+6)(1-x)^2}{12}\Big\} \tag{3.9}$$

$$\lambda_{n+1}(x) = \lambda_0(-x).$$

THEOREM 2. *The four operators* $\Lambda_{n,p}(f; x)$, $p = 0, 1, 2, 3$ (*with* $\omega(x) = T_n(x)$, $P_n(x)$, $U_n(x)$ *and* $P'_{n+1}(x)$, *respectively*) *are positive for n large enough uniformly in* $[-1, 1]$.

Proof. It is clear from (3.7)–(3.9) that $\lambda_0(x)$, $\lambda_{n+1}(x)$ are $\geqslant 0$ in all the four cases considered. It is therefore sufficient from (3.1) and (3.6) to check if the linear function $L_k(x)$ is nonnegative in $[-1, 1]$. The case $p = 0$, $\nu = 1$, $\omega(x) = T_n(x)$ was considered in [5] where it is shown that $L_k(x) \geqslant 0$ for all $n \geqslant 2$. For $p \geqslant 1$ and $\nu \geqslant 2$, we observe that it is enough to check $L_k(x)$ for $k = 1$ and $x = 1$. Then $L_k(1)$ becomes

$$L_1(1) = \frac{\nu - 3}{3} - \frac{\nu(\nu - 2)}{6} x_1 + \frac{2}{3} N(1 - x_1), \tag{3.10}$$

where x_1 is the zero of $\omega(x)$ closest to 1. If $x_1 = \cos\theta_{1n}$, then from ([8]) (p. 192. Theorem 8.1.2)), we have

$$\lim_{n\to\infty} n\theta_{1n} = j_{1\nu},$$

where $j_{1\nu}$ is the first zero of the Bessel function $J_{(\nu-2)/2}(x)$. Then $\lim_{n\to\infty} L_1(1) = ((\nu - 3)/3) - (\nu(\nu - 2))/6) + \frac{1}{3} j_{1\nu}^2$. From the table of zeros of $J_\nu(x)$ (Watson [11], p. 748) we have

$$j_{1,1} = \pi/2 \qquad j_{1,3} = \pi$$
$$j_{1,2} = 2.4048256 \qquad j_{1,4} = 3.8317060.$$

Since the limit is positive, $L_k(x) \geqslant 0$ for n sufficiently large uniformly on $[-1, 1]$. This completes the proof of Theorem 2.

4. CONVERGENCE OF $\Lambda_{n,p}(f; x)$

We have shown that the operator $\Lambda_{n,p}(f; x)$ is nonnegative for large n, if $p = 0, 1, 2$, or 3 and the nodes $\{x_k\}_1^n$ are the zeros of $T_n(x)$, $P_n(x)$, $U_n(x)$ and $P'_{n+1}(x)$, respectively. We shall examine the uniform convergence of $\Lambda_{n,p}(f; x)$ to $f(x)$ as $n \to \infty$ for $|x| \leqslant 1$. More precisely we have Theorem 3.

THEOREM 3. *Let $f(x)$ be a given function continuous on $[-1, 1]$ and let p be an integer, with $p = 0, 1, 2$ or 3. Let the system of nodes $\{x_\nu\}_1^n$ be the zeros of $T_n(x)$, $P_n(x)$, $U_n(x)$ or $P'_{n+1}(x)$ according as $p = 0, 1, 2$ or 3, respectively. Then*

$$\lim_{n\to\infty} A_{n,p}(f; x) = f(x),$$

uniformly in $[-1, 1]$.

For the proof of this theorem we shall need the following lemmas.

LEMMA 1. (Szegö [8] p. 159 (7.21.1)). *For $n \geqslant 4$ and $-1 \leqslant x \leqslant 1$, we have*

$$(1 - x^2)^{1/4} \mid P_n(x) \mid \leqslant (2/n\pi)^{1/2} \tag{4.1}$$

$$(1 - x^2)^{3/4} \mid P_n'(x) \mid \leqslant (2(n + 1))^{1/2}. \tag{4.2}$$

LEMMA 2. *If $x_1, \ldots, x_n$ are the zeros of $P_n(x)$, then we have*

$$\sum_{k=1}^{n} \frac{1}{(1 - x_k^2)^3 (P_n'(x_k))^4} \frac{1}{1 - x_k} < C_1, \tag{4.3}$$

where C_1 is a constant independent of n.

Proof. Since

$$\sum_{k=1}^{n} \lambda_k(x) = 1 - P_n^4(x), \tag{4.4}$$

we have

$$\sum_{k=1}^{n} l_k^4(x) \cdot \frac{(x - x_k)^2 M_k(x)}{(1 - x_k^2)^3} < \frac{1 - P_n^4(x)}{1 - x^2},$$

where $M_k(x) = -\frac{1}{3} + \frac{2}{3} n(n + 1)(1 - xx_k)$. Letting $x \to 1$, we have

$$\sum_{k=1}^{n} \frac{M_k(1)}{(P_n'(x_k))^4 (1 - x_k^2)^3} \cdot \frac{1}{(1 - x_k)^2} < n(n + 1).$$

Since $M_k(1) = -\frac{1}{3} + \frac{2}{3} n(n + 1)(1 - x_k) > C \cdot (n + 1)(1 - x_k)$, for some suitable constant $C > 0$, we have (4.3).

Remark. Inequality (4.3) is stronger than the following:

$$\sum_{k=1}^{n} \frac{1}{(1 - x_k^2)^3 (P_n'(x_k))^4} < 1$$

which follows from (4.4) on equating the coefficients of x^{4n}.

LEMMA 3. *If $x_1, \ldots, x_n$ are the zeros of $P'_{n+1}(x)$, then*

$$\sum_{k=1}^{n} \frac{1}{(1-x_k^2)(P_{n+1}(x_k))^4} \frac{1}{1-x_k} = O(n^4). \tag{4.10}$$

The proof of this lemma uses (3.1) with $p = 3$ and (3.9) instead of (3.7) following the same pattern as that of Lemma 2 and is therefore omitted.

Proof of Theorem 3. From the positivity of the operators $\Lambda_{n,p}(f; x)$ for large n, it is enough to use Korovkin's theorem. Since $\Lambda_{n,p}(f; x) \equiv 1$, we shall show that $\Lambda_{n,p}((x-t)^2; x)$ converges to zero uniformly in $[-1, 1]$. We observe that

$$\Lambda_{n,p}((1-t)^2; +1) = \Lambda_{n,p}((1-t)^2; -1) = 0.$$

We shall now consider the four cases separately.

Case (i). $p = 0$, $\omega(x) = T_n(x)$. Then

$$\Lambda_{n,0}(f; x) = \sum_{k=1}^{n} f(x_k)\, \lambda_k(x),$$

where

$$\lambda_k(x) = \frac{1}{n^4}\left(\frac{T_n(x)}{x-x_k}\right)^4 \cdot \left[(1-xx_k)^2 + (x-x_k)^2 \left\{\frac{2}{3}(n^2-1)(1-xx_k) - \frac{1}{2}xx_k\right\}\right].$$

The convergence for this case was proved in [5].

Case (ii). $p = 1$, $\omega(x) = P_n(x)$. In this case $\nu = 2$ and from (3.1), (3.6) and (3.7) we have after some simplification

$$\begin{aligned} \Lambda_{n,1}((x-t)^2; x) &= (1-x^2)(P_n(x))^4 \\ &\quad + \sum_{k=1}^{n} l_k^4(x) \frac{1-x^2}{1-x_k^2}(x-x_k)^2 \\ &\quad \times \left[\frac{1-2xx_k+x_k^2}{1-x_k^2} + \frac{(x-x_k)^2}{(1-x_k^2)^2} M_k(x)\right] \\ &= S_{n0} + S_{n1} + S_{n2}, \end{aligned}$$

where

$$l_k(x) = \frac{P_n(x)}{(x-x_k)\, P_n'(x_k)}, \qquad M_k(x) = -\frac{1}{3} + \frac{2}{3}n(n+1)(1-xx_k).$$

By Lemma 1, we have

$$S_{n0} = (1 - x^2)(P_n(x))^4 < \frac{C_1}{n^2}, \qquad -1 \leqslant x \leqslant 1. \tag{4.5}$$

Also

$$S_{n1} = \sum_{k=1}^{n} l_k^4(x) \cdot \frac{1 - x^2}{1 - x_k^2} (x - x_k)^2 \cdot \frac{1 - 2xx_k + x_k^2}{1 - x_k^2}$$

$$\leqslant \left(\sum_{k=1}^{n} l_k^2(x) \cdot \frac{1 - x^2}{1 - x_k^2}\right)\left(\sum_{k=1}^{n} l_k^2(x) \cdot \frac{1 - 2xx_k + x_k^2}{1 - x_k^2} \cdot (x - x_k)^2\right).$$

From the quasi-H–F interpolation formula of P. Szasz [7] we have

$$\sum_{k=1}^{n} \frac{1 - x^2}{1 - x_k^2} l_k^2(x) = 1 - P_n^2(x), \tag{4.6}$$

and the H–F interpolation formula (Szegö [8] p. 340) shows that

$$\sum_{k=1}^{n} l_k^2(x) \cdot \frac{1 - 2xx_k + x_k^2}{1 - x_k^2} (x - x_k)^2 \to 0, \tag{4.7}$$

uniformly in each subinterval $[-1 + \epsilon, 1 - \epsilon]$ as $n \to \infty$ and is uniformly bounded in $[-1, 1]$. (4.6) and (4.1) yield that

$$S_{n1} \to 0, \qquad \text{uniformly in } [-1, 1], \tag{4.8}$$

as $n \to \infty$.

Using Lemma 1, we have

$$S_{n2} \leqslant C(1 - x^2)\, P_n^4(x) \cdot n^2 \sum_{k=1}^{n} \frac{1}{(1 - x_k^2)^3\, (P_n'(x_k))^4}$$

$$\leqslant C \sum_{k=1}^{n} \frac{1}{(1 - x_k^2)^3\, (P_n'(x_k))^4}$$

$$= C\left(\sum_{|x_k| \geqslant 1-\epsilon} + \sum_{|x_k| < 1-\epsilon}\right).$$

From Lemma 2, the first sum above is $\leqslant C_{1\epsilon}$. Also ([8] formula (8.9.2) p. 238),

$$P_n'(\cos\theta_k) \sim k^{-3/2} n^2, \qquad x_k = \cos\theta_k$$

so that

$$\sum_{|x_k| < 1-\epsilon} < \frac{C_2}{\epsilon_3} \sum_{k=1}^{n} \frac{k^6}{n^8} < \frac{C_3}{n\epsilon^3}.$$

Hence choosing $\epsilon = n^{-1/4}$, we have

$$S_{n2} \leqslant C_4 \cdot n^{1/4}. \tag{4.9}$$

The proof of Case (ii) is completed on combining (4.5), (4.8) and (4.9).

Case (iii). $p = 2$, $\omega(x) = U_n(x) = (\sin(n+1)\,\theta/\sin\theta)$, $x = \cos\theta$. As in Case (ii) it is enough to consider $\Lambda_{n2}((x-t)^2; x)$. In this case $\nu = 3$, hence from (3.8), (3.1) and (3.6), we have

$$\Lambda_{n2}((x-t)^2; x) = S_{n0} + S_{n1} + S_{n2}$$

where

$$\begin{aligned} S_{n0} &= (x-1)^2\,\lambda_0(x) + (x+1)^2\,\lambda_{n+1}(x) \\ &= \frac{U_n^4(x)(1-x^2)^2}{2(n+1)^4} \cdot \frac{(2n+1)(2n+3)+3}{3} = O\left(\frac{1}{n^2}\right). \end{aligned}$$

$$S_{n1} = \sum_{k=1}^{n} \left\{(x-x_k)^2\,\frac{U_n^2(x)(1-x^2)(1-xx_k)}{(x-x_k)^2\,(n+1)^2}\right\}^2$$

and

$$S_{n2} = \frac{(1-x^2)^2\,U_n^4(x)}{(n+1)^4} \sum_{k=1}^{n} \left\{-\frac{1}{2}\,xx_k + \frac{1}{3}\,n(n+2)(1-xx_k)\right\}.$$

Since $\sum_{k=1}^{n} x_k = 0$, it follows that

$$S_{n2} = (1-x^2)\,U_n^4(x) \cdot O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right).$$

From a result of P. Szasz [7] (p. 426, formula (13)) we know that

$$\sum_{k=1}^{n} \frac{(1-x^2)\,U_n^2(x)(1-xx_k)}{(n+1)^2\,(x-x_k)^2} \leqslant 1$$

so that

$$S_{n1} \leqslant \sum_{k=1}^{n} (x-x_k)^2\,\frac{U_n^2(x)(1-x^2)(1-xx_k)}{(x-x_k)^2\,(n+1)^2} = O\left(\frac{1}{n}\right).$$

This shows that

$$\Lambda_{n2}((x-t)^2; x) = O(1/n). \tag{4.10}$$

Case (iv). $p = 3$, $\omega(x) = cP'_{n+1}(x)$. Set $\Pi_{n+2}(x) = (1-x^2)\,P'_{n+1}(x) = -(n+2)(n+1)\int_{-1}^{x} P_{n+1}(t)\,dt$. In this case $\nu = 4$ and for $k = 1, \ldots, n$

$$\lambda_k(x) = l_k^4(x) + \frac{2}{3}\,\frac{l_k^4(x)}{1-x^2} \cdot \frac{1-xx_k}{1-x_k^2}\,(n+1)(n+2)(x-x_k)^2 \tag{4.11}$$

where

$$l_k(x) = \frac{\Pi_{n+2}(x)}{(x - x_k)\,\Pi'_{n+2}(x_k)}, \tag{4.12}$$

and $\lambda_0(x)$, $\lambda_{n+1}(x)$ are given by (3.9). Then

$$A_{n3}((x - t)^2; x) = S_{n0} + S_{n1} + S_{n2},$$

where

$$S_{n0} = (1 - x)^2 \lambda_0(x) + (1 + x)^2 \lambda_{n+1}(x),$$

$$S_{n1} = \sum_{k=1}^{n} (x - x_k)^2\, l_k^4(x),$$

$$S_{n2} = \frac{2(n + 1)(n + 2)}{3(1 - x^2)} \sum_{k=1}^{n} (x - x_k)^4\, l_k^4(x) \cdot \frac{1 - x x_k}{1 - x_k^2}.$$

From the fact that $P'(x) = O(n^2)$ on $[-1, 1]$ and from Lemma 1 it follows easily that

$$S_{n0} = O(1/n^2). \tag{4.13}$$

From a result of Fejér [3], we have

$$\sum_{k=1}^{n} l_k^2(x) \leqslant 1$$

and from a result of Turán [9], we get

$$S_{n1} \leqslant \sum_{k=1}^{n} (x - x_k)^2\, l_k^2(x) \to 0,$$

uniformly in $[-1, 1]$ as $n \to \infty$. Again using Lemma 1 we have

$$\begin{aligned} S_{n2} &= \frac{2(1 - x^2)^3 (P'_{n+1}(x))^4}{3(n + 1)^3 (n + 2)^3} \sum_{k=1}^{n} \frac{1}{(1 - x_k^2)(P_{n+1}(x_k))^4} \\ &= O\left(\frac{1}{n^4}\right) \sum_{k=1}^{n} \frac{1}{(1 - x_k^2)(P_{n+1}(x_k))^4} \\ &= \sigma_{n1} + \sigma_{n2}, \end{aligned}$$

where

$$\sigma_{n1} = O\left(\frac{1}{n^4}\right) \sum_{|x_k| < 1-\epsilon}, \qquad \sigma_{n2} = O\left(\frac{1}{n^4}\right) \sum_{|x_k| \geqslant 1-\epsilon}.$$

From the differential equation for $P_{n+1}(x)$, we have

$$(1 - x_k^2) P''_{n+1}(x_k) = -(n+1)(n+2) P_{n+1}(x_k), \qquad k = 1,..., n.$$

Hence

$$\sigma_{n1} = O(n^4) \sum_{|x_k|<1-\epsilon} \frac{1}{(1 - x_k^2)^5 (P''_{n+1}(x_k))^4}.$$

Again from Szegö [8] (formula (8.9.2) p. 238), we have

$$P''_{n+1}(x_k) = \left(\frac{d}{dx} P_n^{(3/2)}(x)\right)_{x_k} \sim k^{-5/2} n^4.$$

Then

$$\sigma_{n1} = O(1/n\epsilon^5).$$

Using Lemma 3, we have

$$\sigma_{n2} = O\left(\frac{1}{n^4}\right) \sum_{|x_k|\geqslant 1-\epsilon} \cdot \frac{1}{(1 - x_k^2)(P_{n+1}(x_k))^4}$$

$$\leqslant \epsilon \cdot O\left(\frac{1}{n^4}\right) \sum \frac{1}{(1 - x_k^2)(P_{n+1}(x_k))^4} \frac{1}{1 - x_k} < C\epsilon.$$

Choose $\epsilon = n^{-1/6}$. Then

$$\sigma_{n1} = O(n^{-1/6}), \qquad \text{and} \qquad \sigma_{n2} = O(n^{-1/6}),$$

so that $S_{n2} \to 0$ as $n \to \infty$. This completes the proof of Case (iv).

5. Conclusion

It is possible to obtain analogues of Theorem 2 and 3 by assuming conditions only at $+1$ or at -1 as in [6]. In view of the proof of Theorem 2, it is easy to observe that operators $A_{n,p}(x)$ will be positive also when the x_i's are the zeros of some ultraspherical polynomials not discussed here. However the formulae become more complicated. Also Theorem 3 leads to a problem analogous to that proposed and resolved by Turán [9] in connection with Hermite interpolation. More precisely, we could ask whether the convergence behaviour of the operators $A_{n,p}(f; x)$ will be changed when we do not prescribe the third derivative at a x_k and whether we can find a class of functions for which the convergence persists. For similar results for the case $A_{n,0}$ with $\omega(x) = T_n(x)$ we refer to [5]. We shall return to this problem elsewhere.

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